KMS STATES ON C*-ALGEBRAS ASSOCIATED WITH SELF-SIMILAR SETS

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Abstract. In this paper, we study KMS states for the gauge actions on C*-algebras associated with self-similar sets whose branch points are finite. If the self-similar set does not contain any branch point, the Hutchinson measure gives the unique KMS state. But if the self-similar set dose contain a branch point, there sometimes appear other KMS states which come from branch points. For this purpose we construct explicitly a basis for a Hilbert C*-module associated with a self-similar set with finite branch condition. Using this we get condition for a Borel probability measure on K to be extended to a KMS state on the C*-algebra associated with the original self-similar set. We classify KMS states for the case of dynamics of unit interval and the case of Sierpinski gasket which is related with Complex dynamical system. KMS states for these examples are unique and given by the Hutchinson measure if β is equal to $\log N$, where N is the number of contractions. They are expressed as convex combinations of KMS states given by measures supported on the orbit of the branched points if $\beta > \log N$.

1. Introduction

There exist many interactions between reversible topological dynamical systems and their C*-algebras through the crossed product construction by groups of homeomorphisms on compact Hausdorff spaces. On the other hand we have many interesting examples of irreversible dynamical systems of continuous maps like a tent map on the unit interval and rational functions on the Riemann sphere. They are often branched covering maps or expansive maps on compact metric spaces and their inverse branches sometimes consist of proper contractions. Thus a family of proper contractions on a compact metric space which is self-similar with respect to these contractions give a irreversible dynamical system in some sense. Although C*-algebras of groupoids by Renault [21] and [22] are also useful for irreversible systems, we study C*-algebras of bimodules by Pimsner [19] to include singular points. In our study we show that there exists a relation between the orbit structure of branched points for irreversible dynamical systems and the structure of KMS states for the gauge actions on their C^*-algebras. In this paper we study irreversible systems defined by family of proper contractions on a self-similar set. In [11] we introduced a C*-algebra $\mathcal{O}_\gamma = \mathcal{O}_\gamma(K)$ associated with a system $\gamma = (\gamma_1, \dots, \gamma_N)$ of contractions on a self-similar set K. We explicitly determine the KMS states in the case of contractions corresponding to a tent map

on the unit interval and the rational function $R(z) = \frac{z^3 - \frac{16}{27}}{z}$ on the Julia set J_R, which is homeomorphic to the Sierpinski gasket. Since Sierpinski gasket contains three branched points and their inverse orbits by contractions fall in fixed points, we have that for any $\beta > \log 3$, the set of β-KMS sates is homeomorphic to a two-dimensional simplex spanned by the three vertices corresponding to the three branched points.

We recall that Olsen-Pedersen [18] showed that KMS state with inverse temperature β (i.e.,β-KMS state) on Cuntz algebra $\mathcal{O}_n$ with respect to the gauge action exists if and only if $\beta = \log n$ and that $\log n$-KMS state is unique. Evans [3] extended their result for quasi-free actions. Enomoto-Fujii-Watatani [2] studied the gauge action on Cuntz-Krieger algebras $\mathcal{O}_A$, and show that the KMS state is unique and its inverse temperature is the logarithm of the Perron-Frobenius eigenvalue of A for a irreducible matrix A. Exel-Laca [6] studied KMS states on partial crossed product C*-algebras by free groups and classify KMS states on Cuntz-Krieger algebras associated with infinite matrix. Exel studied KMS states more in [4] and [5]. More generally KMS states on Cuntz-Pimsner algebras are studied by Pinzari-Watatani-Yonetani [20], Kerr and Pinzari [12] and Laca-Neshveyev [14]. Kumjian and Renault [13] investigated KMS states on groupoid C*-algebras associated with expansive maps. In many cases the logarithm of inverse temperature of a KMS state is equal to the entropy of the corresponding dynamical systems like sofic shifts [15], [20]. But except the value of entropy, we have been unable to catch any information of the structure of the dynamical systems from the property of KMS states. The aim of the paper is to get a information on the structure of branched points from the structure of KMS states.

We introduced a C*-algebra $\mathcal{O}_R$ associated with a rational function R in [10] and show that if the Julia set does not contain any branched point, then the Lyubich measure gives the unique $\log \deg R$-KMS state for the gauge action.

In this paper, we study KMS states on C*-algebras associated with self-similar sets whose branched points are finite. If the self-similar set does not contain any branched point, then the Hutchinson measure gives the unique KMS state for the gauge action. But if the self-similar set does contain a branched point, there sometimes appear another KMS state. To study it, we need to construct a concrete countable basis for Hilbert C*-bimodules. First we recall the definition and fundamental results for basis for Hilbert C*-module, which will be used without saying explicitly. The fact that each basis automatically converges unconditionally is important and used in several occasions. Next, we characterize a KMS state on $\mathcal{O}_X$ in terms of its restriction to the coefficient algebra A, where A is a unital C*-algebra and X is a countably generated Hilbert C*-bimodule over A. This comes from a general theorem in Laca-Neshveyev [14]. But they extend traces on A to Toeplitz algebra in some specific class first, extend general traces by perturbation of action and take weak limits. We here provide a simple

and direct proof using the properties of countable basis for clear understanding of extension from traces on A to the fixed point algebra of $\mathcal{O}_X$ by the gauge action. Next, we explicitly construct a basis called a patched basis, for Hilbert C*-modules constructed from self-similar sets which satisfy the finite branch condition. Using a patched basis, we express the condition that tracial states on $A = \mathrm{C}(K)$ extend to KMS states on $\mathcal{O}_\gamma$ without using basis. We obtain some Ruell-Perron-Frobenius like operator concerning the condition that measure on K is extended to KMS states. Last, we classify KMS states for some specific examples. We treat dynamics on unit interval. It is shown that there exists a unique $\log N$-KMS states. This state is of infinite type defined in [14]. When there exists a branched value in K, another type of KMS states appear. For each branched point y, there exists a KMS states which is expressed as a countable sum of Dirac measures. These are of finite type in [14]. The KMS states of these C*-algebra are expressed by convex combinations of them. When a family of contraction is the section of an map h, the minimum value of the logarithm of inverse temperature of KMS states is shown to be the entropy of h. We do similar classification of KMS states for the C*-algebra associated with Sierpinski gasket introduced in [11].

The content of this paper is as follows: In section 2, we recall several definitions and fundamental facts. In section 3, we give a characterization of KMS states on Cuntz-Pimsner algebra using a countable basis. In section 4, we construct basis with the finite branch condition and provide a characterization of KMS states in terms of measures on the self-similar set. In section 5, we present classification results of KMS states for specific examples.

The method presented in this paper to classify KMS states on the C* algebra expressed as a Cuntz-Pimsner algebra with an abelian C*-algebra as a coefficient algebra is applicable to many other cases, and we hope that our method shed light on the role of branched points in Cuntz-Pimsner algebra constructed from correspondences with branches. We shall study KMS states on the C*-algebra associated with complex dynamical systems in the forthcoming papers.

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2. Self-similar sets and Hilbert C* bimodules

Let (K, d) be a compact metric space.

Definition 2.1. *A continuous map γ on K is called a proper contraction if there exists constants $0 < c_1 \leq c_2 < 1$ such that*

$$c_1 d(y_1, y_2) \leq d(\gamma(y_1), \gamma(y_2)) \leq c_2 d(y_1, y_2) \qquad y_1,\ y_2 \in K.$$

Let N be an integer greater than 1, and let $\gamma = (\gamma_1, \dots, \gamma_N)$ be a family of proper contractions on (K, d). We use notations such as $\gamma(x) = \bigcup_{j=1}^N \{\gamma_j(x)\}$ and $\gamma^{-1}(x) = \bigcup_{j=1}^N \{\gamma_j^{-1}(x)\}$.

Definition 2.2. *K is called self-similar with respect to γ when $K = \bigcup_{i=1}^{N} \gamma_i(K)$.*

Lemma 2.3. *If K is self similar with respect to γ, K has no isolated point.*

Proof. We fix $x \in K$. Then for each n there exists $(j_1, \ldots, j_n)$ such that $x \in \gamma_{j_1} \cdots \gamma_{j_n}(K)$ because K is self-similar with respect to γ. Since γ_j's are proper contraction, the diameter of $\gamma_{j_1} \cdots \gamma_{j_n}(K)$, which is homeomorphic to K, tends to zero as $n \to \infty$. This shows that $x \in K$ is not isolated in K. $\square$

We need additional technical conditions.

Definition 2.4. *We say that $\gamma = (\gamma_1, \ldots, \gamma_N)$ satisfies the open set condition if there exists a non empty open subset V of K such that $\bigcup_{i=1}^{N} \gamma_i(V) \subset V$ and $\gamma_j(V) \cap \gamma_{j'}(V) = \phi$ for $j \neq j'$.*

Define subsets $B(\gamma)$, $C(\gamma)$ and $\tilde{C}(\gamma)$ of K by

$$B(\gamma) = \{x \in K | x = \gamma_j(y) = \gamma_{j'}(y) \text{ for some } y \in K \text{ and } j \neq j'\}$$
$$C(\gamma) = \{y \in K | \gamma_j(y) = \gamma_{j'}(y) \text{ for } j \neq j'\}$$
$$\tilde{C}(\gamma) = \bigcup_{j=1}^{N} \gamma_j^{-1}(B(\gamma)).$$

We call a point in $B(\gamma)$ a branched point, and a point in $C(\gamma)$ a branched value.

Definition 2.5. *We say that γ satisfies the finite branch condition if $C(\gamma)$ is a finite set.*

We define a branched index $e(x, y)$ when $x \in \gamma(y)$ by

$$e(x, y) = \#\{j \in \{1, \ldots, N\} | \gamma_j(y) = x\}$$

For $x \in K$ we define $I(x)$ by

$$I(x) = \{j \in \{1, \ldots, N\} | \text{ there exists } y \in K \text{ such that } x = \gamma_j(y)\}.$$

We also use the following notation: For $y \in K$,

$$O(y) = \bigcup_{n=0}^{\infty} \{\gamma_{j_1} \cdots \gamma_{j_n}(y) | (j_1, \cdots, j_n) \in \{1, \cdots, N\}^n\},$$

with a convention $\gamma_{j_1} \cdots \gamma_{j_n}(y) = y$ for $n = 0$. We call $O(y)$ the orbit of y.

Example 2.1. *Let $K = [0, 1]$, $\gamma_1(y) = \frac{1}{2}y$, and $\gamma_2(y) = 1 - \frac{1}{2}y$. Then K is self-similar with respect to $\gamma = (\gamma_1, \gamma_2)$. γ satisfies the open set condition. In this example, we can take $V = (0, 1)$. We have $B(\gamma) = \{\frac{1}{2}\}$ and $C(\gamma) = \{1\}$. We refer this example as the case of tent map.*

Example 2.2. *Let $K = [0, 1]$, $\gamma_1(y) = \frac{1}{2}y$ and $\gamma_2(y) = \frac{1}{2}(y + 1)$. Then K is self-similar with respect to γ. In this case, $B(\gamma) = \phi$ and $C(\gamma) = \phi$.*

We recall Cuntz-Pimsner algebras [19]. Let A be a C*-algebra and X be a Hilbert right A-module. We denote by $L(X)$ the algebra of the adjointable bounded operators on X. For $\xi,\ \eta \in X$, the "rank one" operator $\theta_{\xi,\eta}$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)$ for $\zeta \in X$. The closure of the linear span of rank one operators is denoted by $K(X)$. We say that X is a Hilbert bimodule over A if X is a Hilbert right A- module with a *-homomorphism $\phi : A \to L(X)$. We always assume that X is full and ϕ is injective. Let $F(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$ be the full Fock module of X with a convention $X^{\otimes 0} = A$. For $\xi \in X$, the creation operator $T_\xi \in L(F(X))$ is defined by

$$T_\xi(a) = \xi a \qquad \text{and} \ \ T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

We define $i_{F(X)} : A \to L(F(X))$ by

$$i_{F(X)}(a)(b) = ab \qquad \text{and} \ \ i_{F(X)}(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a, b \in A$. The Cuntz-Toeplitz algebra $\mathcal{T}_X$ is the C*-algebra on $F(X)$ generated by $i_{F(X)}(a)$ with $a \in A$ and T_ξ with $\xi \in X$. Let $j_K|K(X) \to \mathcal{T}_X$ be the homomorphism defined by $j_K(\theta_{\xi,\eta}) = T_\xi T_\eta^*$. We consider the ideal $I_X := \phi^{-1}(K(X))$ of A. Let $\mathcal{J}_X$ be the ideal of $\mathcal{T}_X$ generated by $\{i_{F(X)}(a) - (j_K \circ \phi)(a); a \in I_X\}$. Then the Cuntz-Pimsner algebra $\mathcal{O}_X$ is defined as the quotient $\mathcal{T}_X/\mathcal{J}_X$. Let $\pi : \mathcal{T}_X \to \mathcal{O}_X$ be the quotient map. Put $S_\xi = \pi(T_\xi)$ and $i(a) = \pi(i_{F(X)}(a))$. Let $i_K : K(X) \to \mathcal{O}_X$ be the homomorphism defined by $i_K(\theta_{\xi,\eta}) = S_\xi S_\eta^*$. Then $\pi((j_K \circ \phi)(a)) = (i_K \circ \phi)(a)$ for $a \in I_X$. We note that the Cuntz-Pimsner algebra $\mathcal{O}_X$ is the universal C*-algebra generated by $i(a)$ with $a \in A$ and S_ξ with $\xi \in X$ satisfying that $i(a)S_\xi = S_{\phi(a)\xi}$, $S_\xi i(a) = S_{\xi a}$, $S_\xi^* S_\eta = i((\xi|\eta)_A)$ for $a \in A$, $\xi, \eta \in X$ and $i(a) = (i_K \circ \phi)(a)$ for $a \in I_X$. We usually identify $i(a)$ with a in A. We also identify S_ξ with $\xi \in X$ and simply write ξ instead of S_ξ. We denote by $\mathcal{O}_X^{alg}$ the *-algebra generated algebraically by A and X. There exists an action $\alpha : \mathbb{R} \to \mathrm{Aut}\ \mathcal{O}_X$ with $\alpha_t(\xi) = e^{it}\xi$, which is called the gauge action. Since we assume that $\phi : A \to L(X)$ is isometric, there is an embedding $\phi_n : L(X^{\otimes n}) \to L(X^{\otimes n+1})$ with $\phi_n(T) = T \otimes id_X$ for $T \in L(X^{\otimes n})$ with a convention $\phi_0 = \phi : A \to L(X)$. There exists an isometric map $j_K^{(n)}$ from $K(X^{\otimes n})$ to $\mathcal{O}_X$ such that $j_K^{(n)}(\theta_{\xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \cdots \otimes \eta_n}) = \xi_1 \cdots \xi_n {\eta_n}^* \cdots {\eta_1}^*$. We also identify $K(X^{\otimes n})$ and its image in $\mathcal{O}_X$.

We put

$$\tilde{\mathcal{F}}_n = \phi_{n-1} \circ \cdots \circ \phi_1 \circ \phi(A) + \phi_{n-1} \circ \cdots \circ \phi_1(K(X)) + \cdots + K(X^{\otimes n}).$$

with a convention $\tilde{\mathcal{F}}_0 = A$. We denote by $\mathcal{F}_n$ the C*-algebra generated by A, $K(X)$, $\cdots$, and $K(X^{\otimes n})$ in $\mathcal{O}_X$ with a convention $\mathcal{F}_0 = A$. Then there exists a family of isomorphisms $\{\Psi_n\}_{n=0}^{\infty}$ between $\tilde{\mathcal{F}}_n$ and $\mathcal{F}_n$, which is compatible with two filtrations $\tilde{\mathcal{F}}_i \subset \tilde{\mathcal{F}}_{i+1}$ and $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ [19] and [7]. The algebra $\mathcal{F}_X = \overline{\bigcup_{n=0}^{\infty} \mathcal{F}_n}$ coincides with the fixed point algebra under the gauge action α. We denote by E the conditional expectation from $\mathcal{O}_X$ to $\mathcal{F}_X$ given by α.

Let (K, d) be a compact metric space, $\gamma = (\gamma_1, \cdots, \gamma_N)$ be a system of proper contraction. We put $\mathcal{C}_\gamma = \bigcup_{j=1}^N \{(\gamma_j(y), y) | y \in K\}$, $A = \mathrm{C}(K)$ and $X_\gamma = \mathrm{C}(\mathcal{C}_\gamma)$. For a, $b \in A$ and f, $g \in X$, we define two A actions and right A-inner product on X as follows:

$$(a \cdot f \cdot b)(x, y) = a(x) f(x, y) b(y) \qquad (f|g)_A(y) = \sum_{j=1}^N \overline{f(\gamma_j(y), y)} g(\gamma_i(y), y).$$

Then X_γ is a full Hilbert A-module. If we put $\phi(a) f = a \cdot f$, then ϕ is an isometric *-homomorphism from A to $L(X)$. Then X_γ is a full Hilbert C*-bimodule over A.

Definition 2.6. *The Cuntz-Pimsner algebra $\mathcal{O}_{X_\gamma}$ constructed from X_γ is denoted by $\mathcal{O}_\gamma = \mathcal{O}_\gamma(K)$.*

Theorem 2.7. [11] *When γ satisfies the open set condition, $\mathcal{O}_\gamma$ is simple and purely infinite.*

Lemma 2.8. *If γ satisfies the finite branch condition, we have $I_X = \{a \in A \,|\, a(y) = 0 \quad for\ y \in B(\gamma)\}$.*

Proof. This lemma is proved in Proposition 2.4 [11] under the open set condition. But the open set condition is only used to take a sequence $(x_n, y_n) \in \mathcal{C}_\gamma$ such that $x_n \notin B(\gamma)$, $x_n \to c$ and $y_n \to d$ for (c, d) with $c \in B(\gamma)$ and $d \in C(\gamma)$. When K is self similar with respect to γ and γ satisfies the finite branch condition, by Lemma 2.3 these sequence always exists. □

This lemma also follows from Theorem 3.11 in [17].

For $x \in X$, we write $\|x\|_2 = \|(x|x)_A\|^{1/2}$. We note that if $|f_n(\gamma_j(y), y) - f(\gamma_j(y), y)| \to 0$ uniformly with respect to y for every j, then we have $\|f_n - f\|_2 \to 0$.

We recall bases for Hilbert C*-modules [9]. In the following, we assume that A is σ-unital and X is countably generated. A family $(u_\lambda)_{\lambda \in \Lambda}$ in X indexed by a set Λ, which is countable or finite, is called a basis for X if for every $\varepsilon > 0$ there exists a finite subset $F_0 \subset \Lambda$ such that for every finite subset F such that $F \supset F_0$ we have

$$\left\| \sum_{k \in F} u_k (u_k | f)_A - f \right\| < \varepsilon,$$

that is, a net $\{\sum_{k \in F} u_k (u_k|f)_A | F \subset \Lambda \text{ finite subset } \}$ converges to f with respect to $\| \cdot \|$. We write this as

$$\sum_{k \in \Lambda} u_k (u_k | f)_A = f.$$

We note that for countably infinite Λ an index set $(u_k)_{k\in\Lambda}$ is a basis if and only if for any numbering of Λ we have

$$\sum_{k=1}^{\infty} u_k(u_k|f)_A = f$$

in norm. This means that the series converges to f unconditionally.

We assume that a countable sequence $\{u_k\}_{k=1}^{\infty}$ in X satisfies

$$\sum_{k=1}^{\infty} u_k(u_k|f)_A = f$$

for $f \in X$. Then by Proposition 1.2 in [9], An indexed family $(u_k)_{k\in\mathbf{N}}$ is automatically a basis for X, where $\mathbf{N}$ denotes the set of natural numbers.

We have the following lemmas.

Lemma 2.9. *Let $(u_k^i)_{k\in\Lambda^i}$, $(i = 1,\dots,n)$ be families in X indexed by countable sets Λ^i. Put $\Lambda = \bigcup_{i=1}^n \{(i,k)|k \in \Lambda^i\}$. We assume that for every $f \in X$,*

$$f = \sum_{i=1}^{n} \sum_{k\in\Lambda^i} u_k^i(u_k^i|f)_A.$$

Then the indexed family $(u_k^i)_{(i,k)\in\Lambda}$ is a basis for X.

Lemma 2.10. *Let $(u_k)_{k\in\Lambda}$ be a basis for X. Then $(u_{k_1}\otimes\cdots\otimes u_{k_n})_{(k_1,\dots,k_n)\in\Lambda^n}$ is a basis for $X^{\otimes n}$.*

3. KMS states on Pimsner algebras

Let $\mathcal{A}$ be a C*-algebra with a one parameter automorphism group α. A state φ on $\mathcal{A}$ is called a β-KMS state with respect to α if

$$\varphi(a\alpha_{i\beta}(b)) = \varphi(ba)$$

holds for $a \in \mathcal{A}$ and $b \in \mathcal{A}_a$, where $\mathcal{A}_a$ denotes the set of entire analytic elements for α in $\mathcal{A}$. We refer for the definition and fundamental matters of KMS states to Bratteli-Robinson [1].

Let I be an ideal of a C*-algebra B. Let ψ be a positive linear functional of on I. The natural extension $\tilde{\psi}$ of ψ to B is given by $\psi(b) = \lim_\lambda \psi(be_\lambda)$ for $b \in B$, where $(e_\lambda)_\lambda$ is an approximate unit in I. We need the following general lemma.

Lemma 3.1. *(Proposition 12.5 [6]) Let B be a unital C*-algebra. Suppose $B = A + I$ where A is a C*-subalgerba containing 1 and I is a closed two sided ideal. Let ϕ be a state on A and ψ be a positive linear functional on I. We denote by $\tilde{\psi}$ the natural extension of ψ to B. Then if (1) $\phi \geq \tilde{\psi}$ on A^+ and (2) $\phi = \psi$ on $A\cap I$, there exists a state ρ on B such that $\rho|_A = \phi$ and $\rho|_I = \psi$. Moreover, such a state ρ is unique.*

Let A be a unital C*-algebra, X be a countably generated full Hilbert A-module and ϕ be an injective *-homomorphism from A to $L(X)$. Let $\mathcal{O}_X$ be a Cuntz-Pimsner algebra constructed from X. We also need the following lemma.

Lemma 3.2. *(Lemma 4.2(2) [7]) We have $\mathcal{F}^{(n-1)} \cap K(X^{\otimes n}) = K(X^{\otimes n-1}) \cap K(X^{\otimes n})$ in $\mathcal{O}_X$.*

The following proposition follows from the general theorem in [14]. But since the extension procedure in [14] of traces on A to $\mathcal{O}_X$ is not straightforward using perturbation, we give a simple and direct proof within $\mathcal{O}_X$, which is a natural extension of Lemma 3.2 in [20]. We fix a basis $(u_k)_{k\in\Lambda}$. We assume that Λ is countably infinite, and admit $u_k = 0$ for some $k \in \Lambda$. Let $\{u_k\}_{k=1}^{\infty}$ denote the sequence obtained from $(u_k)_{k\in\Lambda}$ by some numbering of Λ.

Proposition 3.3. *The restriction of a β-KMS state φ on $\mathcal{O}_X$ to A is a tracial state τ on A satisfying the following conditions:*

$$\sum_{k=1}^{\infty} \tau((u_k|au_k)_A) = \lambda\tau(a) \quad (\forall a \in I_X) \quad (1)$$

$$\sum_{k=1}^{\infty} \tau((u_k|au_k)_A) \leq \lambda\tau(a) \quad (\forall a \in A^+) \quad (2)$$

for $\lambda = e^{\beta}$.

For a tracial state τ on A satisfying (1) and (2), we can construct a β-KMS state φ on $\mathcal{O}_X$ whose restriction to A coincides with τ. Moreover, such an extension is unique.

Proof. Let φ be a β-KMS state on $\mathcal{O}_X$. By the condition of β-KMS state, we have

$$\sum_{k=1}^{n} \varphi(u_k^* a u_k) = \lambda \sum_{k=1}^{n} \varphi(a u_i u_i^*)$$

where $a \in A$. Since $\sum_{k=1}^{n} u_k u_k^* \leq I$ and $\varphi|_{\mathcal{O}_X^{(0)}}$ is a trace, we have

$$0 \leq \lambda \sum_{k=1}^{n} \varphi(a u_k u_k^*) = \lambda\varphi\left(a^{1/2}\left(\sum_{k=1}^{n} u_k u_k^*\right) a^{1/2}\right) \leq \lambda\varphi(a)$$

for $a \in A^+$. Then we have,

$$\sum_{k=1}^{\infty} \varphi(u_k^* a u_k) \leq \lambda\varphi(a).$$

Let $a \in I_X \subset \mathcal{K}(X)$. Since $(\sum_{k=1}^{n} \theta_{u_k,u_k})_{n=1}^{\infty}$ is an approximate unit in $K(X)$, we have

$$\sum_{k=1}^{\infty} \varphi(u_k^* a u_k) = \lambda \lim_{n\to\infty} \varphi(a \sum_{k=1}^{n} \theta_{u_k,u_k}) = \lambda\varphi(a).$$

Then we conclude that (1) and (2) hold.

We take a tracial state τ on A satisfying the condition (1) and (2). Let F_n be a finite subset of $\mathbf{N}^n$. We define finite sets $F_{p-1}, \cdots, F_1$ by $F_{p-1} = \{(k_1, \dots, k_{p-1}) \in \mathbf{N}^{p-1} | (k_1, \dots, k_{p-1}, k_p) \in F_p\}$ inductively. $(2 \le p \le n)$. Then using the condition (2) repeatedly, we have

$$
\begin{aligned}
&\lambda^{-n} \sum_{(k_1,\dots,k_n)\in F_n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | u_{k_1} \otimes \cdots \otimes u_{k_n})_A) \\
=&\lambda^{-n} \sum_{(k_1,\dots,k_n)\in F_n} \tau(u_{k_n}^* \cdots u_{k_1}^* u_{k_1} \cdots u_{u_n}) \\
\le&\lambda^{-n} \sum_{k_n=1}^{\infty} \sum_{(k_1,\dots,k_{n-1})\in F_{n-1}} \tau(u_{k_n}^* u_{k_{n-1}}^* \cdots u_{k_1}^* u_{k_1} \cdots u_{k_{n-1}} u_{k_n}) \\
\le&\lambda^{-n+1} \sum_{(k_1,\dots,k_{n-1})\in F_{n-1}} \tau(u_{k_{n-1}}^* \cdots u_{k_1}^* u_{k_1} \cdots u_{u_{n-1}}) \\
\le&\lambda^{-p} \sum_{(k_1,\dots,k_p)\in F_p} \tau(u_{k_{p-1}}^* \cdots u_{k_1}^* u_{k_1} \cdots u_{k_{p-1}}) \\
\le&1.
\end{aligned}
$$

Since $x \le \|x\| I$ for $x \in L(X^{\otimes n})^+$, we have

$$
\lambda^{-n} \sum_{(k_1,\dots,k_n)\in F_n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | x u_{k_1} \otimes \cdots \otimes u_{k_n})_A) \le \|x\|.
$$

This shows that $\sum_{(k_1,\dots,k_n)\in\Lambda^n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | x u_{k_1} \otimes \cdots \otimes u_{k_n})_A)$ converges unconditionally for each $x \in L(X^{\otimes n})$. Then we can define a bounded positive linear functional σ^n on $L(X^{\otimes n})$ by

$$
\sigma^n(x) = \lambda^{-n} \sum_{(k_1,\dots,k_n)\in\mathbf{N}^n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | x u_{k_1} \otimes \cdots \otimes u_{k_n})_A).
$$

We put $\tau^n = \sigma^n|_{K(X^{\otimes n})}$. For $x_1 \otimes \cdots \otimes x_n$, $y_1 \otimes \cdots \otimes y_n \in X^{\otimes n}$, we have

$$
\begin{aligned}
&\tau^n(\theta_{x_1\otimes\cdots\otimes x_n, y_1\otimes\cdots\otimes y_n}) \\
=&\lambda^{-n} \sum_{(k_1,\ldots,k_n)\in\mathbf{N}^n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | \theta_{x_1\otimes\cdots\otimes x_n, y_1\otimes\cdots\otimes y_n}(u_{k_1} \otimes \cdots \otimes u_{k_n}))_A) \\
=&\lambda^{-n} \sum_{(k_1,\cdots,k_n\in)\mathbf{N}^n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | x_1 \otimes \cdots \otimes x_n(y_1 \otimes \cdots \otimes y_n | u_{k_1} \otimes \cdots \otimes u_{k_n})_A)_A) \\
=&\lambda^{-n} \sum_{(k_1,\cdots,k_n)\in\mathbf{N}^n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | x_1 \otimes \cdots \otimes x_n)_A(y_1 \otimes \cdots \otimes y_n | u_{k_1} \otimes \cdots \otimes u_{k_n})_A) \\
=&\lambda^{-n} \sum_{(k_1,\cdots,k_n)\in\mathbf{N}^n} \tau((y_1 \otimes \cdots \otimes y_n | u_{k_1} \otimes \cdots \otimes u_{k_n})_A(u_{k_1} \otimes \cdots \otimes u_{k_n} | x_1 \otimes \cdots \otimes x_n)_A) \\
=&\lambda^{-n} \sum_{(k_1,\cdots,k_n)\in\mathbf{N}^n} \tau((u_{k_1} \otimes \cdots \otimes u_{k_n}(u_{k_1} \otimes \cdots \otimes u_{k_n} | y_1 \otimes \cdots \otimes y_n)_A | x_1 \otimes \cdots \otimes x_n)_A) \\
=&\lambda^{-n}\tau(\sum_{(k_1,\cdots,k_n)\in\mathbf{N}^n} (u_{k_1} \otimes \cdots \otimes u_{k_n}(u_{k_1} \otimes \cdots \otimes u_{k_n} | y_1 \otimes \cdots \otimes y_n)_A | x_1 \otimes \cdots \otimes x_n)_A) \\
=&\lambda^{-n}\tau((y_1 \otimes \cdots \otimes y_n | x_1 \otimes \cdots \otimes x_n)_A),
\end{aligned}
$$

because an indexed set $(u_{k_1} \otimes \cdots \otimes u_{k_n})_{(k_1,\ldots,k_n)\in\Lambda^n}$ is a basis for $X^{\otimes n}$ by Lemma 2.10.

Let F be a finite subset of Λ^n. We put $e_F = \sum_{\{k_1,\cdots,k_n\}\in F} \theta_{u_{k_1}\otimes\cdots\otimes u_{k_n}, u_{k_1}\otimes\cdots\otimes u_{k_n}}$. Then $\{e_F\}_{F \text{ finite subset of } \Lambda^n}$ is an approximate unit in $K(X^{\otimes n})$. The we have

$$
\begin{aligned}
\tilde{\tau}^n(x) &= \lim_F \tau^n(xe_F) \\
&= \lim_F \tau^n(\sum_{(k_1,\ldots,k_n)\in F} \theta_{xu_{k_1}\otimes\cdots\otimes u_{k_n}, u_{k_1}\otimes\cdots\otimes u_{k_n}}) \\
&= \lim_F \tau((u_{k_1} \otimes \cdots \otimes u_{k_n} | xu_{k_1} \otimes \cdots \otimes u_{k_n})_A) \\
&= \sigma^n(x).
\end{aligned}
$$

Thus the natural extension $\tilde{\tau}^n$ to $L(X^{\otimes n})$ is given by σ^n. Since $\tilde{\mathcal{F}}^{(n-1)} \subset L(X^{\otimes n}) = M(K(X^{\otimes n}))$, the natural extension $\tilde{\tau}^n$ of τ^n to $\mathcal{F}^{(n-1)}$ is given by $\sigma^n \circ \Psi_n^{-1}$.

For each $n \geq 0$, we define states ω^n on $\mathcal{F}^{(n)}$ which extend τ such that $\omega^{n+1}|_{\mathcal{F}^{(n)}} = \omega^n$ and $\omega^n|_{K(X^{\otimes n})} = \tau^n$. First, we put $\omega^0 = \tau$. Then ω^0 is a state on $\mathcal{F}^{(0)} = A$. We assume $n \geq 1$, and assume that there exist states ω^i on $\mathcal{F}^{(i)}$ for $0 \leq i \leq n$ such that $\omega^i|_{K(X^{\otimes i})} = \tau^i$ for $1 \leq i \leq n$ and $\tilde{\tau}^i \leq \omega^i$ on $\mathcal{F}^{(i-1)}$ for $1 \leq i \leq n$. We have $\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + K(X^{\otimes n+1})$, and by Lemma 3.2 we have $\mathcal{F}^{(n)} \cap K(X^{\otimes n+1}) = K(X^{\otimes n}) \cap K(X^{\otimes n+1})$. Let $x \in K(X^{\otimes n}) \cap K(X^{\otimes n+1})$. Using

$u_{i_n}^* \cdots u_{i_1} x u_{i_1} \cdots u_{i_n} \in A \cap K(X) = I_X$ and the condition (1), we have

$$\begin{aligned}\tau^{n+1}(x) &= \lambda^{-n-1} \sum_{(i_1,\cdots,i_n,i_{n+1})\in \mathbf{N}^{n+1}} \tau(u_{i_{n+1}}^* u_{i_{i_n}}^* \cdots u_{i_1}^* x u_{i_1} \cdots u_{i_n} u_{i_{n+1}}) \\ &= \lambda^{-n} \sum_{(i_1,\cdots,i_n)\in \mathbf{N}^{n}} \tau(u_{i_{i_n}}^* \cdots u_{i_1}^* x u_{i_1} \cdots u_{i_n}) \\ &= \tau^n(x) = \omega^n(x).\end{aligned}$$

For $x \in \mathcal{F}^{(n)}$ we write $x = y + z$ where $y \in \mathcal{F}^{(n-1)}$ and $z \in K(X^{\otimes n})$. By the assumption of induction, we have $\tilde{\tau}^n(y^*y) \le \omega^n(y^*y)$. Since $y^*z + z^*y + z^*z \in K(X^{\otimes n})$, we have $\tau^n(y^*z + z^*y + z^*z) = \omega^n(y^*z + z^*y + z^*z)$. We note that for $x \in {\mathcal{F}^{(n)}}^+$ we have

$$\begin{aligned}\tilde{\tau}^{n+1}(x) &= \lambda^{-n-1} \sum_{(i_1,\dots,i_n,i_{n+1})\in \mathbf{N}^{n+1}} \tau(u_{i_{n+1}}^* u_{i_n}^* \cdots u_{i_1}^* x u_{i_1} \cdots u_{i_n} u_{i_{n+1}}) \\ &\le \lambda^{-n} \sum_{(i_1,\dots,i_n)\in \mathbf{N}^{n+1}} \tau(u_{i_n}^* \cdots u_{i_1}^* x u_{i_1} \cdots u_{i_n}) \\ &= \tilde{\tau}^n(x).\end{aligned}$$

We used the fact that $u_{i_n}^* \cdots u_{i_1}^* x u_{i_1} \cdots u_{i_n} \in A^+$ because $x \in {\mathcal{F}^{(n)}}^+$ and the condition (2). Then, we have

$$\begin{aligned}\tilde{\tau}^{n+1}(x^*x) &= \tilde{\tau}^{n+1}((y+z)^*(y+z)) \\ &\le \tilde{\tau}^n((y+z)^*(y+z)) \\ &= \tilde{\tau}^n(y^*y + y^*z + z^*y + z^*z) \\ &= \tilde{\tau}^n(y^*y) + \tau^n(y^*z + z^*y + z^*z) \\ &= \tilde{\tau}^n(y^*y) + \omega^n(y^*z + z^*y + z^*z) \\ &\le \omega^n(y^*y) + \omega^n(y^*z + z^*y + z^*z) \\ &= \omega^n(y^*y + y^*z + z^*y + z^*z) \\ &= \omega^n(x^*x).\end{aligned}$$

By Lemma 3.1, there exists a state ω^{n+1} on $\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + K(X^{\otimes n+1})$ such that $\omega^{n+1}|_{\mathcal{F}^{(n)}} = \omega^n$, and $\omega^{n+1}|_{K(X^{\otimes n+1})} = \tau^{n+1}$. Moreover we have $\tilde{\tau}^{n+1}|_{\mathcal{F}^{(n)}} \le \omega^{(n+1)}$.

By induction, we have states ω^n on $\mathcal{F}^{(n)}$ for all $n \ge 0$ such that $\omega^{n+1}|_{\mathcal{F}^{(n)}} = \omega^n$, and $\omega^n|_{K(X^{\otimes n})} = \tau^n$. Putting $\omega(x) = \omega^n(x)$ for $x \in \mathcal{F}^{(n)}$, we define ω on $\bigcup_{i=0}^{\infty} \mathcal{F}^{(i)}$. Since ω^n is a state on $\mathcal{F}^{(n)}$, we can extend ω to a state on the closure $\mathcal{F}_X$.

We note that ω does not depend on the choice of a basis $(u_k)_{k\in\Lambda}$ on X because the values of ω on $K(X^{\otimes n})$ are expressed without basis.

Let m be an integer. We show $\omega(xy^*) = \lambda^m \omega(y^*x)$ for x, $y \in \mathcal{O}_X^{(m)}$. It is sufficient to prove this equality for nonnegative m, and when x and y are in the

form

$$x = z_1 \dots z_m x_1 \cdots x_p y_p^* \cdots y_1^*, \quad y = z_1' \cdots z_m' x_1' \cdots x_q' {y_q'}^* \cdots {y_q'}^*$$

where z_i, z_i', x_j, x_j', y_k, $y_k' \in X$. Put

$$a = (y_p^* \cdots y_1^*)(y_1' \cdots y_p') \in A, \quad \tilde{a} = ({x_p'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^*)(z_1 \cdots z_m x_1 \cdots x_p) \in A.$$

We may assume that $p \leq q$. Another case is similar. We have

$$\begin{aligned}
&\omega(xy^*) \\
=&\omega(z_1 \dots z_m x_1 \cdots x_p y_p^* \cdots y_1^* y_1' \cdots y_q' {x_q'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^*) \\
=&\omega(z_1 \dots z_m x_1 \cdots x_p (y_p^* \cdots y_1^* y_1' \cdots y_p') y_{p+1}' \cdots y_q' {x_q'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^*) \\
=&\omega(z_1 \dots z_m x_1 \cdots x_p a y_{p+1}' \cdots y_q' {x_q'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^*) \\
=&\lambda^{-(q+m)} \tau({x_q'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^* z_1 \dots z_m x_1 \cdots x_p a y_{p+1}' \cdots y_q') \\
=&\lambda^{-(q+m)} \tau({x_q'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^* z_1 \dots z_m x_1 \cdots x_p (y_p^* \cdots y_1^* y_1' \cdots y_p') y_{p+1}' \cdots y_q') \\
=&\lambda^{-m} \lambda^{-q} \tau({x_q'}^* \cdots {x_{p+1}'}^* ({x_p'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^* z_1 \cdots z_m x_1 \cdots x_p) y_p^* \cdots y_1^* y_1' \cdots y_q') \\
=&\lambda^{-m} \lambda^{-q} \tau({x_q'}^* \cdots {x_{p+1}'}^* \tilde{a}^* y_p^* \cdots y_1^* y_1' \cdots y_q') \\
=&\lambda^{-m} \omega(y_1' \cdots y_q' {x_q'}^* \cdots {x_{p+1}'}^* \tilde{a}^* y_p^* \cdots y_1^*) \\
=&\lambda^{-m} \omega(y_1' \cdots y_q' {x_q'}^* \cdots {x_{p+1}'}^* ({x_p'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^* z_1 \cdots z_m x_1 \cdots x_p) y_p^* \cdots y_1^*) \\
=&\lambda^{-m} \omega(y_1' \cdots y_q' {x_q'}^* \cdots {x_1'}^* {z_m'}^* \cdots {z_1'}^* z_1 \cdots z_m x_1 \cdots x_p y_p^* \cdots y_1^*) \\
=&\lambda^{-m} \omega(y^* x).
\end{aligned}$$

When $m = 0$, this shows that ω is a trace. When $m = 1$, this shows that ω can be extended to a β-KMS state φ on $\mathcal{O}_X$ by $\varphi = \omega \circ E$. $\square$

If Λ is finite, we have $I_X = A$ and the condition (2) is unnecessary. In this case, this proposition is Lemma 3.2 of [20]. If (2) holds, $\sum_{k=1}^{\infty} \tau((u_k|au_k)_A)$ dose not depend on the numbering of Λ and we can write $\sum_{k \in \Lambda} \tau((u_k|au_k)_A)$.

4. KMS states on a C*-algebra associated with a self-similar set

Let (K, d) be a compact metric space and $\gamma = (\gamma_1, \cdots, \gamma_N)$ be a set of proper contractions on K. In the following, we assume that γ satisfies the finite branch condition. We construct a basis for Hilbert C*-module X_γ over A.

For this purpose we consider the following situation. Let K_1 and K_2 be compact metric spaces, n be an integer and γ_i $(i = 1, \cdots, n)$ be proper contractions from K_1 to K_2. For $n \geq 2$, we assume that there exists a $c \in K_1$ such that $\gamma_1(c) = \cdots = \gamma_n(c)$ and for $\gamma_i(y)$'s are different $y \neq c$. We put $\mathcal{C} = \{(\gamma_i(y), y) | y \in K_1,\ i = 1, \dots, n\}$, $A = \mathrm{C}(K_1)$ and $X = \mathrm{C}(\mathcal{C})$. Then X is a right Hilbert A module. We say that such a module is of n-branch class. We construct a basis for Hilbert C*-module of n-branch class. If $n = 1$, we put $\Lambda = \{1\}$ and $u_1(x, y) = 1$. Then $(u_k)_{k \in \Lambda}$ is a basis for X.

Assume that $n \geq 2$. We fix a positive P. We define a family of a functions $r_i(x)$ on $[0, \infty)$ for $i \geq 1$ by

$$r_i(x) = \begin{cases} 1 \quad \frac{P}{i} \leq x \\ \left(\frac{i}{2P}\right) x - 1 \quad \frac{P}{2i} \leq x \leq \frac{P}{i} \\ 0 \quad 0 \leq x \leq \frac{P}{2i} \end{cases}$$

and with a convention $r_0(x) = 0$. For $i \geq 0$, $r_i(x)$ is a non decreasing function and $r_i(x) \leq r_{i+1}(x)$ for every x. We put $v_i(x) = (r_i(x) - r_{i-1}(x))^{1/2}$ for $i \geq 1$. Let $\delta > 0$, then there exists an $i_\delta > 0$ such that $\sum_{\tilde{i}=1}^{i_\delta} v_{\tilde{i}}(\delta)^2 = 1$. For $x \geq \delta$ and $i \geq i_\delta$ we have $v_i(x) = 0$, and then we have $\sum_{\tilde{i}=1}^{i} v_{\tilde{i}}(x)^2 = 1$.

Let $\omega = e^{2\pi\sqrt{-1}/n}$. Since $\gamma_j(y)$ $(j = 1, \cdots, n)$ are different for $y \neq c$ and $v_i(0) = 0$, we can do the following definition. For $k \geq 1$, we define a family of continuous functions u_k in X as follows:

$$u_1(x, y) = \frac{1}{\sqrt{n}}$$

$$u_{1+(n-1)(i-1)+l}(\gamma_j(y), y) = \frac{1}{\sqrt{n}}\, \omega^{lj}\, v_i(d(y, c)),$$

where $i \geq 1$ and $1 \leq l \leq n - 1$.

For y with $d(y, c) \geq \delta$ and k with $k \geq 1 + (n - 1)(i - 1)$ where $i \geq i_\delta$, we have $u_k(\gamma_j(y), y) = 0$ for each j. Let $M \geq 1 + (n - 1)(i_\delta - 1)$, and put $f_M = \sum_{k=1}^{M} u_k(u_k|f)_A$. For each j, we have

$$\begin{aligned} f_M(\gamma_j(y), y) &= \sum_{k=1}^{M} u_k(\gamma_j(y), y) \sum_{\tilde{j}=1}^{n} \overline{u_k(\gamma_{\tilde{j}}(y), y)} f(\gamma_{\tilde{j}}(y), y) \\ &= \frac{1}{n} \sum_{\tilde{j}=1}^{n} f(\gamma_{\tilde{j}}(y), y) + \frac{1}{n} \sum_{\tilde{i}=1}^{i_\delta} \sum_{\tilde{j}=1}^{n} \sum_{l=1}^{n-1} \omega^{l(j-\tilde{j})} v_{\tilde{i}}(d(y, c))^2 f(\gamma_{\tilde{j}}(y), y) \\ &= \frac{1}{n} \sum_{\tilde{j}=1}^{n} f(\gamma_{\tilde{j}}(y), y) + \frac{(n-1)}{n} \sum_{\tilde{i}=1}^{i_\delta} v_{\tilde{i}}(d(y, c))^2 f(\gamma_j(y), y) \\ &- \frac{1}{n} \sum_{\tilde{j}=1, (\tilde{j} \neq j)}^{n} \sum_{\tilde{i}=1}^{i_\delta} v_{\tilde{i}}(d(y, c))^2 f(\gamma_{\tilde{j}}(y), y) \\ &= f(\gamma_j(y), y). \end{aligned}$$

We used $\sum_{l=1}^{n-1} \omega^{l(\tilde{j}-j)}$ is -1 for $\tilde{j} \neq j$ and $n-1$ for $\tilde{j} = j$, and $\sum_{\tilde{i}=1}^{i_\delta} v_{\tilde{i}}(d(y, c))^2 = 1$ for y with $\delta \leq d(y, c)$.

We take arbitrary small $\varepsilon' > 0$. There exists a $\delta > 0$ satisfying the following: If $d(y, c) < \delta$, then we have

$$|f(\gamma_j(y), y) - f(b, c)| < \varepsilon'$$

for each j. We write as $M = 1 + (n-1)i + l$ with $i \geq 1$ and $1 \leq l \leq n-1$. For $1 \leq j \leq n$ we have

$$\begin{aligned}
&|f_M(\gamma_j(y), y) - f(\gamma_j(y), y)| \\
=&|f_M(\gamma_j(y), y) - f(b, c)| + |f(\gamma_j(y), y) - f(b, c)| \\
\leq&|f_M(\gamma_j(y), y) - f(b, c)| + \varepsilon'.
\end{aligned}$$

We estimate $|f_M(\gamma_j(y), y) - f(b, c)|$.

$$\begin{aligned}
|f_M(\gamma_j(y), y) - f(b, c)| \leq & \left| \frac{1}{n} \sum_{\tilde{j}=1}^{n} (f(\gamma_{\tilde{j}}(y), y)) - f(b, c) \right| \\
&+ \left| \frac{1}{n} \sum_{p=1}^{n-1} \sum_{\tilde{j}=1}^{n} \omega^{p(\tilde{j}-j)} \sum_{\tilde{i}=1}^{i} v_{\tilde{i}}(d(y, c))^2 (f(\gamma_{\tilde{j}}(y), y) - f(b, c)) \right| \\
&+ \left| \frac{1}{n} \sum_{p=1}^{n-1} \sum_{\tilde{j}=1}^{n} \omega^{p(\tilde{j}-j)} \sum_{\tilde{i}=1}^{i} v_{\tilde{i}}(d(y, c))^2 f(b, c) \right| \\
&+ \left| \frac{1}{n} \sum_{p=1}^{l} \sum_{\tilde{j}=1}^{n} \omega^{p(\tilde{j}-j)} v_{i+1}(d(y, c))^2 (f(\gamma_{\tilde{j}}(y), y) - f(b, c)) \right| \\
&+ \left| \frac{1}{n} \sum_{p=1}^{l} \sum_{\tilde{j}=1}^{n} \omega^{p(\tilde{j}-j)} v_{i+1}(d(y, b))^2 f(b, c) \} \right| \\
\leq & \frac{1}{n} n\varepsilon' + \frac{1}{n} n(n-1) \left(\sum_{\tilde{i}=1}^{i} v_{\tilde{i}}(d(y, b))^2 \right) \varepsilon' + \frac{1}{n} lnv_{i+1}(d(y, b))\varepsilon' \\
\leq & \varepsilon' + (n-1) \left(\sum_{\tilde{i}=1}^{i+1} v_{\tilde{i}}(d(y, c))^2 \right) \varepsilon' \\
\leq & n\varepsilon'.
\end{aligned}$$

We used $\sum_{\tilde{j}=1}^{n} \omega^{p(\tilde{j}-j)} = 0$ for $1 \leq p \leq n-1$ and $\sum_{\tilde{i}=1}^{i+1} v_{\tilde{i}}(d(y, c))^2 \leq 1$. We take arbitrary small ε, and take ε' with $0 < \varepsilon' < \varepsilon/(1+n)$ and choose $\delta > 0$ for such an ε'. We choose sufficiently large M_0 such that for every $M \geq M_0$, $f_M(\gamma_j(y), y) = f(\gamma_j(y), y)$ holds for every $d(y, c) \geq \delta$ and for every j. We can conclude that for every $M \geq M_0$,

$$|f_M(\gamma_j(y), y) - f(\gamma_j(y), y)| < \varepsilon$$

for all $y \in K$. We have shown the following proposition.

Proposition 4.1. *If a right Hilbert C^*-module X is of n-branch class and $n \geq 2$, $(u_k)_{k\in\mathbf{N}}$ as above is a basis for X.*

We note that if c is an accumulation point in K, $(u_k)_{k\in\Lambda}$ is actually a countably infinite basis.

Since $\# C(\gamma)$ is finite, we put $C(\gamma) = \{c_1, \ldots, c_m\}$, where $c_i \neq c_{i'}$ for $i \neq i'$. We take sufficiently small open neighborhoods U_i of c_i such that $C(\gamma) \cap \overline{U}_i = \{c_i\}$ for $1 \leq i \leq m$ and $\overline{U}_i \cap \overline{U}_{i'} = \phi$ for $i \neq i'$. We take an open neighborhood V_i of c_i such that $\overline{V}_i \subset U_i$ for each i. We put $U_{m+1} = K \setminus \bigcup_{i=1}^m \overline{V}_i$. Then $\{U_i\}_{i=1}^{m+1}$ is an open covering of K.

We put $A_i = \mathrm{C}(\overline{U}_i)$, $\mathcal{C}_i = \{(x,y) \in \mathcal{C} | y \in \overline{U}_i\} \subset \mathcal{C}$ and $X_i = \mathrm{C}(\mathcal{C}_i)$, $i = 1, \cdots, m+1$. Then X_i's are right Hilbert A_i module naturally. We fix a $c_i \in C(\gamma)$. We put $\{b_1^i, \ldots, b_{r_i}^i\} = \gamma(c_i)$, where b_s^i and $b_{s'}^i$ are different for $s \neq s'$. We put $\mathcal{C}_{i,s} = \{(x,y) \in \mathcal{C}_i | x = \gamma_k(y) \text{ for some } k \in I(b_s^i)\}$. Then we have $\mathcal{C}_i = \bigcup_{s=1}^{r_i} \mathcal{C}_{i,s}$ and $\mathcal{C}_{i,s} \cap \mathcal{C}_{i,s'} = \phi$ for $s \neq s'$. We put $X_{i,s} = \mathrm{C}(\mathcal{C}_{i,s})$. Then $X_{i,s}$'s are right Hilbert A_i module and we have

$$(X_i)_{A_i} = \bigoplus_{s=1}^{r_i} (X_{i,s})_{A_i}.$$

Let $(u_k^{i,s})_{k\in\Lambda^{i,s}}$ be the basis for $X_{i,s}$ defined in Proposition 4.1. Put $\Lambda^i = \bigcup_{s=1}^{r_i} \{(s,k) | k \in \Lambda^{i,s})\}$.

Lemma 4.2. *An indexed family $(u_k^{i,s})_{(s,k)\in\Lambda^i}$ is a basis for X_i for each i.*

Proof. Since X_i is a direct sum of $X_{i,s}$'s, we can conclude the lemma by Lemma 2.9. □

Let $\{\psi_i\}_{i=1}^{m+1}$ be a partition of unity associated with the open covering $\{U_i\}_{i=1}^{m+1}$. Let $(u_k^{i,s})_{(s,k)\in\Lambda^i}$ be a basis for X_i given by Lemma 4.2. We put $\tilde{u}_k^{i,s}(\gamma_j(y), y) = u_k^{i,s}(\gamma_j(y), y)\psi_i(y)^{1/2}$. Then $\tilde{u}_k^{i,s}$'s can be extended to a function on K. Put $\Lambda = \bigcup_{i=1}^{m+1} \{(i,(s,k)) | (s,k) \in \Lambda^i\}$.

Theorem 4.3. *Let (K,d) be a compact metric space, $\gamma = (\gamma_1, \ldots, \gamma_N)$ be a system of proper contractions on K. We assume that γ satisfies the finite branch condition. Then $(\tilde{u}_k^{i,s})_{(i,(s,k))\in\Lambda}$ defined above is a basis for X.*

Proof. Let $f \in X$. Since $\mathrm{supp}(f \cdot \psi_i) \subset U_i$,

$$(f \cdot \psi_i)(\gamma_j(y), y) = \sum_{(k,s)\in\Lambda^i} u_k^{i,s} (u_k^{i,s} | f \cdot \psi_i)_A (\gamma_j(y), y).$$

uniformly. Using this equation we have

$$\begin{aligned}\sum_{i=1}^{m+1}\sum_{(s,k)\in\Lambda^i}\tilde{u}_k^{i,s}(\tilde{u}_k^{i,s}|f)_A(\gamma_j(y),y) =&\sum_{i=1}^{m+1}\sum_{(s,k)\in\Lambda^i}u_k^{i,s}\cdot\psi_i^{1/2}(u_k^{i,s}\cdot\psi_i^{1/2}|f)_A(\gamma_j(y),y)\\ =&\sum_{i=1}^{m+1}\sum_{(s,k)\in\Lambda^i}u_k^{i,s}(u_k^{i,s}|f\cdot\psi_i)(\gamma_j(y),y)\\ =&\sum_{i=1}^{m+1}(f\cdot\psi_i)(\gamma_j(y),y)\\ =&f(\gamma_j(y),y)\end{aligned}$$

uniformly with respect to y. ☐

Definition 4.4. *We call such basis as constructed in Theorem 4.3 a patched basis for X.*

We put $\gamma_j^*(a)(y) = a(\gamma_j(y))$. For $a \in A$, we define a Borel function $\tilde{a}$ by

$$\tilde{a}(y) = \sum_{x\in\gamma(y)} a(x) = \sum_{j=1}^{N}\frac{1}{e(\gamma_j(y),y)}a(\gamma_j(y)).$$

We note that if $C(\gamma)$ is not empty, $\tilde{a}$ is not continuous.

Lemma 4.5. *Let X be an n-branch module. Then for the basis $(u_k)_{k\in\mathbf{N}}$ constructed in Proposition 4.1 we have*

$$\sum_{k=1}^{\infty}(u_k|au_k)_A(y) = \tilde{a}(y)$$

for every $y \in K$. The left hand side converges unconditionally.

Proof. If $n = 1$, we have

$$(u_1|au_1)_A(y) = a(\gamma_1(y)).$$

We assume $n \geq 2$. Then we have

$$\begin{aligned}\sum_{k=1}^{\infty}(u_k|au_k)_A(y) =&\sum_{k=1}^{\infty}\sum_{j=1}^{n}\overline{u_k(\gamma_j(y),y)}a(\gamma_j(y))u_k(\gamma_j(y),y)\\ =&\sum_{k=1}^{\infty}\sum_{j=1}^{n}|u_k(\gamma_j(y),y))|^2a(\gamma_j(y)))\\ =&\frac{1}{n}\sum_{j=1}^{n}a(\gamma_j(y)) + \frac{n-1}{n}\sum_{i=1}^{\infty}\sum_{j=1}^{n}v_i(d(\gamma_j(y),y))^2a(\gamma_j(y)).\end{aligned}$$

The last expression is equal to $\sum_{j=1}^{n} a(\gamma_j(y))$ if $y \neq c$ and equal to $\frac{1}{n}\sum_{j=1}^{n} a(\gamma_j(y))$ if $y = c$. In any case, this is equal to $\tilde{a}(y)$. If $a \in A^+$, the left hand side is monotone convergent, so we conclude that the left hand side converges unconditionally for general $a \in A$. □

Since the left hand side in Lemma 4.5 converges unconditionally, we may write

$$\sum_{k\in\Lambda}(u_k|au_k)_A(y) = \tilde{a}(y).$$

We take a base $(u_k^{i,s})_{k\in\Lambda^{i,s}}$ as in Theorem 4.3. Then $\sum_{s=1}^{r_i}\sum_{k\in\Lambda^{i,s}}(u_k^{i,s}|au_k^{i,s})_A(y)$ is $\sum_{s=1}^{r_i} a(b_s^i)$ for $y = c_i$ and $\sum_{s=1}^{r_i}(\sum_{j\in I(b_s^i)} a(\gamma_j(y))$ for $y \neq s_i$. This is equal to $\sum_{x\in\gamma(y)} a(x)$ in any case. We take a partition of unity $\{\psi_i\}_{i=1}^{m+1}$ and $(\tilde{u}_k^{i,s})_{(i,(s,k))\in\Lambda}$ as in Theorem 4.3. We have

$$\begin{aligned}\sum_{i=1}^{m+1}\sum_{s=1}^{r_i}\sum_{k\in\Lambda^{i,s}}(\tilde{u}_k^{i,s}|a\tilde{u}_k^{i,s})_A(y) &= \sum_{i=1}^{m+1}\psi_i(y)\sum_{s=1}^{r_i}(u_k^{i,s}|au_k^{i,s})_A(y)(y)\\ &= \sum_{i=1}^{m+1}\psi_i(y)\sum_{x\in\gamma(y)} a(x)\\ &= \sum_{x\in\gamma(y)} a(x)\end{aligned}$$

Then we have the following proposition.

Proposition 4.6. *We assume that γ satisfies the finite branch set condition. For a patched basis $(u_k)_{k\in\Lambda}$ for X_γ, we have $\sum_{k\in\Lambda}(u_k|au_k)_A(y) = \tilde{a}(y)$. We note that the left side is monotone convergent for a positive $a \in A$.*

Let $a \in I_X$. Then $a(x) = 0$ for $x \in B(\gamma)$. Since $e(x, y) = 1$ for $x \neq B(\gamma)$, we have $\tilde{a}(y) = \sum_{j=1}^{N} a(\gamma_j(y)) = \sum_{j=1}^{N}\gamma_j^*(a)(y)$.

For a probability measure μ on (K, d), τ^μ denotes the corresponding tracial state on A. For a bounded Borel function a on K, we may define $\tau^\mu(a)$ by $\int_K a(y)d\mu(y)$.

Theorem 4.7. *Let (K, d) be a self-similar set with respect to γ, and γ satisfy the finite branch condition. Let φ be a β-KMS state on $\mathcal{O}_\gamma$. Then the Borel probability measure μ on (K, d) corresponding to the restriction of φ to A satisfies the the following (3) and (4).*

$$\sum_{j=1}^{N}\tau^\mu(\gamma_j^*(a)) = \lambda\tau^\mu(a) \quad (\forall a|_{B(\gamma)} = 0) \quad (3)$$

$$\tau^\mu(\tilde{a}) \leq \lambda\tau^\mu(a) \quad (\forall a \in A^+) \quad (4),$$

where $\lambda = e^\beta$.

Let μ be a probability measure on (K, d) satisfying (3) and (4). Then we can construct a β-KMS state φ on $\mathcal{O}_\gamma$ whose restriction to A is τ^μ. Moreover such a φ is unique.

Proof. Proposition 4.6, remark after Proposition 4.6 and the monotone convergence theorem in integration theory show the theorem. $\square$

In the following, we present results which hold for general situation.

Lemma 4.8. *If a Borel probability measure μ on K satisfies $\sum_{j=1}^N \tau^\mu(\gamma_j^*(a)) = \lambda\tau^\mu(a)$ for arbitrary $a \in A$, a positive constant λ must be equal to N.*

Proof. Putting $a = 1$, we have $\lambda = N$. $\square$

Lemma 4.9. *(Hutchinson [8]) Let (K, d) be a compact metric space and γ be a system of proper contraction. Then there exists a unique measure μ on (K, d) such that*

$$\sum_{j=1}^{N} \tau^\mu(\gamma_j^*(a)) = N\tau^\mu(a)$$

for arbitrary $a \in A$.

Definition 4.10. *We denote by μ^H the measure given by Lemma 4.9, and call this measure the Hutchinson measure.*

Then τ^{μ^H} can be extended to a $\log N$-KMS sate φ^H on $\mathcal{O}_\gamma$.

Remark 4.1. *φ^H is a KMS state of infinite type defined in Laca and Neshveyev [14]. When there exists no branched value for γ, τ^{μ^H} is the unique KMS state on $\mathcal{O}_\gamma$.*

Definition 4.11. *Let μ be a Borel probability measure on (K, d). We define $c_\mu(x)$ by $c_\mu(x) = \mu(\{x\})$. When $c_\mu(x) > 0$, we say that μ has a point mass at x.*

We note that $c_\mu(x)$ is given by $\tau^\mu(\chi_{\{x\}})$ where $\chi_{\{x\}}$ is the characteristic function on a singleton $\{x\}$.

Lemma 4.12. *Let $x \in K$ and assume $\gamma^{-1}(x) = \{y_1, \dots, y_p\}$ with $y_i \neq y_j$ for $i \neq j$. If $x \notin B(\gamma)$, we have $c_\mu(y_1) + \cdots + c_\mu(y_p) = \lambda c_\mu(x)$. If $x \in B(\gamma)$, we have $c_\mu(y_1) + \cdots + c_\mu(y_p) \leq \lambda c_\mu(x)$.*

Proof. Let $x \in K$. We take $\{a_n\}_{n=1}^\infty$ such that $a_n \in A$ and $a_n(x)$ tends to $\chi_{\{x\}}$ pointwise decreasingly. For each j, we have $\gamma_j^*(a_n)$ tends to $\chi_{\{\gamma_j^{-1}(x)\}}$ pointwise decreasingly if $j \in I(x)$, and tend to zero pointwise decreasingly otherwise. We have $\tau^\mu(a_n) \to c_\mu(x)$, if $j \in I(x)$ $\tau^\mu(\gamma_j^*(a_n)) \to c_\mu(\gamma_j^{-1}(x))$ and otherwise $\tau^\mu(\gamma_j^*(a_n)) \to 0$. If $x \notin B(\gamma)$ we can take a_n such that a_n vanish on $B(\gamma)$ for all

n.

$$\lambda c_\mu(x) = \lambda \lim_{n\to\infty} \tau^\mu(a_n) = \lim_{n\to\infty} \sum_{j\in I(x)} \tau^\mu(\gamma_j^*(a_n))$$
$$= \sum_{j\in I(x)} c_\mu(\gamma_j^{-1}(x)) = c_\mu(y_1) + \cdots + c_\mu(y_p).$$

If $x \in B(\gamma)$, we have

$$\lambda c_\mu(x) = \lambda \lim_{n\to\infty} \tau^\mu(a_n) \geq \lim_{n\to\infty} \tau^\mu(\tilde{a_n})$$
$$= \sum_{j\in I(x)} \frac{1}{e(x, \gamma_j^{-1}(x))} c_\mu(\gamma_j^{-1}(x))$$
$$= c_\mu(y_1) + \cdots + c_\mu(y_p).$$

□

Lemma 4.13. *Let μ be a Borel probability measure on (K, d). If μ does not have a point mass at $B(\gamma)$ and satisfies (3) and (4) in Theorem 4.7, we can conclude that $\lambda = N$ and $\mu = \mu^H$. If μ does not have a point mass at $B(\gamma) \cup \tilde{C}(\gamma)$ and satisfies (3), we get the same conclusion.*

Proof. We assume that μ satisfies (4) and does not have a point mass at $B(\gamma)$. We assume that $b \in B(\gamma)$ and $c_1, \dots, c_s$ are mutually different elements in $\tilde{C}(\gamma)$ such that $b = \gamma_{j_1}(c_1) = \gamma_{j_2}(c_2) = \cdots = \gamma_{j_s}(c_s)$. Then by Lemma 4.12 we have

$$0 \leq \sum_{p=1}^{s} c_\mu(c_{j_p}) \leq \lambda c_\mu(b).$$

We conclude that $c_\mu(c_{j_p}) = 0$ for $1 \leq p \leq s$. Since $c_\mu(x) = 0$ for each $x \in B(\gamma)$, we have $c_\mu(y) = 0$ for each $y \in \tilde{C}(\gamma)$.

We assume that μ satisfies (3) and does not have a point mass at $B(\gamma) \cup \tilde{C}(\gamma)$. For each $a \in A^+$, there exists a monotone increasing sequence of $\{a_n\}_{n=1}^\infty \in A$ such that $a_n(x) = 0$ for $x \in B(\gamma)$, and for $x \neq B(\gamma)$ there exits n_0 such that $a_n(x) = a(x)$ for $n \geq n_0$. Then we have $a - a_n$ tends to $\sum_{x\in B(\gamma)} a(x)\chi_{\{x\}}$ and $\gamma_j^*(a) - \gamma_j^*(a_n)$ tends to $\sum_{x\in(B(\gamma)\cap\gamma_j(K))} a(\gamma_j^{-1}(x))\chi_{\{\gamma_j^{-1}(x)\}}$. Since μ has no point mass on $B(\gamma) \cup \tilde{C}(\gamma)$, we have

$$\lim_{n\to\infty} \tau^\mu(a_n) = \tau^\mu(a) \qquad \lim_{n\to\infty} \tau^\mu(\gamma_j^*(a_n)) = \tau^\mu(\gamma_j^*(a)).$$

Then (3) holds for each $a \in A^+$. By Lemma 4.8 and Lemma 4.9, we have $\lambda = N$ and $\mu = \mu^H$. □

5. Classification of KMS states for specific examples

In this section, we present classifications of KMS states for some specific examples.

5.1. **Dynamics on unit interval.** Let $K = [0,1]$, $d(x,y) = |x-y|$ and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions such that K is self similar with respect to γ. The following Lemma is easily verified.

Lemma 5.1. *We assume that γ satisfies the open set condition. For each $x \in B(\gamma)$ and $y \in C(\gamma)$ such that $x \in \gamma(y)$, we have $e(x,y) = 2$. Moreover $C(\gamma) = \tilde{C}(\gamma)$, and they are contained in $\{0,1\}$. $B(\gamma)$ does not contain 0 and 1.*

We always take $(0,1)$ as V. We note that we have $B(\gamma) \subset V$. We may assume that $\{\gamma_j\}_{j=1}^N$ satisfies $\gamma_1(1/2) < \gamma_2(1/2) < \dots < \gamma_N(1/2)$.

In the following we assume that γ satisfies the open set condition.

Lemma 5.2. *Let $y \in B(\gamma)$. Then $O(y) \cap C(\gamma) = \phi$. Let y and y' be distinct points in $B(\gamma)$. Then $O(y) \cap O(y') = \phi$.*

Proof. Since $B(\gamma) \subset V$, $O(y)$ is contained in V. We have $O(y) \cap \{0,1\} = \phi$. We suppose that $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_n}(y) = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_m}(y')$ for y, $y' \in B(\gamma)$. We assume that $n = m$. Then we have $i_1 = j_1$, $i_2 = j_2$ and $i_n = j_n$. We have $y = y'$ and this is a contradiction. We assume that $n \geq m+1$. Then we have $y = \gamma_{n+1}\cdots\gamma_m(y')$, and this is a contradiction. □

Lemma 5.3. *We assume that μ satisfies the conditions (3) and (4) in Theorem 4.7. When $\lambda > 1$, we have $c_\mu(0) = 0$ and $c_\mu(1) = 0$.*

Proof. We suppose that $\gamma_1(0) = 0$ and $\gamma_N(0) = 1$. Since we have $\gamma^{-1}(0) = \gamma_1^{-1}(0) = \{0\}$, we have $c_\mu(0) = \lambda c_\mu(0)$ by Lemma 4.12. Since $\lambda > 1$ we have $c_\mu(0) = 0$. Since we have $\gamma^{-1}(1) = \gamma_N^{-1}(1) = \{0\}$, we have $c_\mu(0) = \lambda c_\mu(1)$ by Lemma 4.12. Then $c_\mu(1) = 0$ follows.

We suppose $\gamma_1(1) = 0$ and $\gamma_N(0) = 1$. Then by a similar computation, we have $c_\mu(1) = \lambda c_\mu(0)$ and $c_\mu(0) = \lambda c_\mu(1)$. Since $\lambda > 1$, we have $c_\mu(0) = 0$ and $c_\mu(1) = 0$.

For other two cases, we can prove lemma similarly. □

Lemma 5.4. *We assume that μ satisfies (3) and (4) in Theorem 4.7 and has a point mass at some point $y \in B(\gamma)$. Then we have $\lambda > N$ and $c_\mu(\gamma_{j_1}\cdots\gamma_{j_n}(y)) = \lambda^{-n}c_\mu(y)$ for each $(j_1, \dots, j_n) \in \{1, \dots, N\}^n$.*

Proof. If $x = \gamma_j(y) = \gamma_{j'}(y')$ for y, $y' \in V$ and $1 \leq j$, $j' \leq N$, we have $y = y'$ and $j = j'$, and for $y \in B(\gamma)$ $\gamma_{j_1}\cdots\gamma_{j_n}(y)$ is contained in $V \cap B(\gamma)^c$ for every $(j_1, \dots, j_n)$, $n \geq 1$. Then by Lemma 4.12, we have

$$c_\mu(\gamma_{j_n}\cdots\gamma_{j_1}(y)) = \lambda c_\mu(\gamma_{j_{n+1}}\gamma_{j_n}\cdots\gamma_{j_1}(y)).$$

for $n \geq 0$. Then we have

$$c_\mu(\gamma_{j_n}\cdots\gamma_{j_1}(y)) = \lambda^{-n}c_\mu(y)$$

By Lemma 5.2 $O(y) \cap C(\gamma) = \phi$. Then

$$\mu(1) \geq \sum_{n=1}^{\infty} \sum_{(j_1,\dots,j_n)\in\{1,\cdots,N\}^n} c_\mu(\gamma_{j_1}\cdots\gamma_{j_n}(y)) = \sum_{n=1}^{\infty} \left(\frac{N}{\lambda}\right)^n c_\mu(y).$$

This shows that $N < \lambda$ is necessary for μ to be bounded. $\square$

Lemma 5.5. *We assume that μ is a Borel probability measure on $[0,1]$ and satisfies (3) and (4) in Theorem 4.7. Then we have $\lambda \geq N$ and $c_\mu(0) = c_\mu(1) = 0$.*

Proof. If μ does not have a point mass at $B(\gamma)$, by Lemma 4.13 we have $c_\mu(y) = 0$ for $y \in C(\gamma) = \tilde{C}(\gamma)$, and have $\mu = \mu^H$ and $\lambda = N$. If μ has a point mass at $B(\gamma)$, then by Lemma 5.4 we have $\lambda > N$. By Lemma 5.5, we have $c_\mu(0) = c_\mu(1) = 0$. $\square$

Lemma 5.6. *A probability measure μ on $[0,1]$ satisfying (3) and (4) in Theorem 4.7 can have point mass only at $\{O(y) | y \in B(\gamma)\}$. In particular, μ^H has no point mass.*

Proof. Let $y \notin \{O(y) | y \in B(\gamma)\}$ and $y \neq 0, 1$. Then we can construct a sequence $\{y_i\}_{i=0}^\infty$ such that $y_0 = y$ and $y_i = \gamma_{j_i}(y_{i+1})$. There exist three possibilities.

(1) All $\{y_0, y_1, \cdots, y_i, \cdots\}$ are different and $y_i \in V \cap B(\gamma)^c$ for all i.
(2) There exists i_0 and $m \geq 1$ such that $y_0, \cdots y_{i_0}$ are all different, and $y_{i_0+m} = y_{i_0}$ and $y_i \in V \cap B(\gamma)^c$ for all i.
(3) There exists i_0 such that $y_0, \cdots y_{i_0-1}$ are all different and in $V \cap B(\gamma)^c$ and $y_{i_0} = 0$ or 1.

In case (1), we have $c_\mu(y_{i+1}) = \lambda c_\mu(y_i)$. Then we have $c_\mu(y_i) = \lambda^i c_\mu(y)$. This shows that if $c_\mu(y) > 0$ $c_\mu(y_i) \to \infty$, and is a contradiction. In case (2), we have $c_\mu(y_{i_0}) = c_\mu(y_{i_0+m}) = \lambda^m c_\mu(y_{i_0})$. Since $\lambda > 1$, we have $c_\mu(y_0) = 0$. $c_\mu(y) = \lambda^{-i_0} c_\mu(y_{i_0})$ shows that $c_\mu(y) = 0$. In case (3), we have $c_\mu(0) + c_\mu(1) = \lambda c_\mu(y_{i_0-1})$, $c_\mu(0) = \lambda c_\mu(y_{i_0-1})$ or $c_\mu(1) = \lambda c_\mu(y_{i_0-1})$. This shows that $c_\mu(y_{i_0-1}) = 0$. Then we have $c_\mu(y) = \lambda^{-i_0} c_\mu(y_{i_0-1}) = 0$. In any case, we can conclude $c_\mu(y) = 0$. $\square$

Let $y \in B(\gamma)$. We consider

$$a \in A^+ \to \sum_{n=0}^{\infty} \sum_{(j_1,\cdots,j_n) \in \{1,\ldots,N\}^n} (1/\lambda)^n a(\gamma_{j_1} \cdots \gamma_{j_n}(y)).$$

This gives a bounded Borel measure on $[0,1]$ if and only if $\lambda > N$. When $\lambda > N$, we can define a Borel probability measure $\mu_{y,\lambda}$ by

$$\mu_{y,\lambda} = \frac{\lambda - N}{\lambda} \sum_{n=0}^{\infty} \sum_{(j_1,\cdots,j_n) \in \{1,\ldots,N\}^n} (1/\lambda)^n \delta_{\gamma_{j_1} \cdots \gamma_{j_n}(y)}$$

for $a \in A$. We put $\beta = \log \lambda$.

Proposition 5.7. *The measure $\mu_{y,\lambda}$ satisfies the condition (3) and (4) in Theorem 4.7, and is extended to a β-KMS state $\varphi^{y,\lambda}$ on $\mathcal{O}_\gamma$.*

Proof. Let $y \in B(\gamma)$. Since

$$\tau^{\mu_{y,\lambda}}(a) = \frac{\lambda - N}{\lambda} \sum_{n=0}^{\infty} \sum_{(j_1,\cdots,j_n)\in\{1,\cdots,N\}^n} \lambda^{-n} a(\gamma_{j_1}\cdots\gamma_{j_n}(y)),$$

we have

$$\lambda\tau^{\mu_{y,\lambda}}(a) = \frac{\lambda - N}{\lambda} \sum_{n=0}^{\infty} \sum_{(j_1,\cdots,j_n)\in\{1,\ldots,N\}^n} \lambda^{-n+1} a(\gamma_{j_1}\cdots\gamma_{j_n}(y)).$$

Since by Lemma 5.2 $\gamma_{j_1}\cdots\gamma_{j_n}(y)$ are not contained in $C(\gamma)$ for every $n \geq 0$, we have

$$\begin{aligned}
\tau^{\mu_{y,\lambda}}(\tilde{a}) =& \frac{\lambda - N}{\lambda} \sum_{n=0}^{\infty} \sum_{(j_1,\cdots,j_n)\in\{1,\cdots,N\}^n} \lambda^{-n} \tilde{a}(\gamma_{j_1}\cdots\gamma_{j_n}(y)) \\
=& \frac{\lambda - N}{\lambda} \sum_{j=1}^{N}\sum_{n=0}^{\infty} \sum_{(j_1,\cdots,j_n)\in\{1,\cdots,N\}^n} \lambda^{-n} a(\gamma_j\gamma_{j_1}\cdots\gamma_{j_n}(y)) \\
=& \frac{\lambda - N}{\lambda} \sum_{n=1}^{\infty} \sum_{(\tilde{j}_1,\cdots,\tilde{j}_n)\in\{1,\cdots,N\}^n} \lambda^{-(n-1)} a(\gamma_{\tilde{j}_1}\cdots\gamma_{\tilde{j}_n}(y)).
\end{aligned}$$

We have

$$\lambda\tau^{\mu_{y,\lambda}}(a) = \tau^{\mu_{y,\lambda}}(\tilde{a}) + \left(\frac{\lambda - N}{\lambda}\right) a(y).$$

If $a \in I_X$, we have $\lambda\tau^{\mu_{y,\lambda}}(a) = \tau^{\mu_{y,\lambda}}(\tilde{a})$ because $a(y) = 0$. If a vanish on $B(\gamma)$, we have $a(y) \geq 0$ and have $\tau^{\mu_{y,\lambda}}(\tilde{a}) \leq \lambda\tau^{\mu_{y,\lambda}}(a)$. □

Remark 5.1. *$\varphi^{y,\lambda}$ is a KMS state of finite type defied in Laca and Neshveyev* [14].

Theorem 5.8. *Let γ be a system of proper contractions on $[0, 1]$ and satisfy the open set condition. Then a β-KMS state on $\mathcal{O}_\gamma$ with respect to the gauge action exists only if $\lambda = e^\beta \geq N$ and are classified as follows:*

(1) *When $\lambda = N$, φ^H is the unique KMS state.*
(2) *When $\lambda > N$, β-KMS state is expressed by a convex combination of $\{\varphi^{y,\lambda} | y \in B(\gamma)\}$.*

Moreover φ^H is unique $\log N$-KMS state, and if $B(\gamma)$ is not empty, $\varphi_{y,\lambda}$ is an extreme $\log\lambda$-KMS state.

Proof. Let μ be a Borel probability measure on $[0, 1]$ and satisfy the condition (3) and (4) in Theorem 4.7. Then by Lemma 5.5, we have $\mu \geq N$. If $\lambda = N$ then μ dose not have point mass at $B(\gamma)$, and then by Lemma 4.13 we have $\mu = \mu^H$.

We assume that $\lambda > N$. By Lemma 5.4 and Lemma 5.3, $\mu - \sum_{y\in B(\gamma)} c_\mu(y)\mu_{y,\lambda}$ is a positive Borel measure, satisfies the condition (3) and does not have point

mass at $B(\gamma) \cup \tilde{C}(\gamma)$. Then by Lemma 4.13, the condition (3) must hold for all $a \in A$. Then we have $\mu - \sum_{y \in B(\gamma)} c_\mu(y)\mu_{y,\lambda} = 0$.

Lastly, we show that $\varphi^{y,\lambda}$ is extreme. We write $\varphi^{y,\lambda} = t\varphi_1 + (1-t)\varphi_2$, where φ_1 and φ_2 be a β-KMS state on $\mathcal{O}_\gamma$ and $0 < t < 1$. By restricting to A, we conclude that $\varphi_i = \varphi^{y,\lambda}$. This shows that $\varphi^{y,\lambda}$ is extreme. □

We assume that γ is a section of a expansive map γ^{-1} on K. The value of inverse temperature of KMS states have a relation with entropy of γ^{-1}.

Proposition 5.9. *The minimum value of the logarithm of the inverse temperature of KMS states on $\mathcal{O}_\gamma$ is equal to the entropy of the map γ^{-1} on K.*

Proof. By Theorem 7.2 in [16], the entropy $h(\gamma^{-1})$ is equal to $\log N$. This is equal to the minimum value of the logarithm of the inverse temperature of KMS states on $\mathcal{O}_\gamma$. □

We consider the example 2.1 i.e. $K = [0,1]$, $\gamma_1(y) = \frac{1}{2}y$, and $\gamma_2(y) = 1 - \frac{1}{2}y$. We denote by $\mathcal{O}_{\text{tent}}$ the Cuntz-Pimsner C*-algebra for this example. We note that $B(\gamma) = \{\frac{1}{2}\}$ and $C(\gamma) = \{1\}$. Let μ be the normalized Lebesgue measure on $[0,1]$. We assume $\lambda > 2$. We put

$$\mu_{1/2,\lambda} = \frac{\lambda - 2}{\lambda} \sum_{n=0}^{\infty} \sum_{(j_1,\cdots,j_n)\in\{1,2\}^n} \lambda^{-n} \delta_{\gamma_{j_1}\cdots\gamma_{j_n}(1/2)}$$

Then we have the following:

Proposition 5.10. *A β-KMS state on $\mathcal{O}_{tent}$ exist if and only if $\beta = \log\lambda \geq \log 2$. If $\lambda = 2$, β-KMS state is unique and given by φ^μ. If $\lambda > 2$, β-KMS state is unique and given by $\varphi^{1/2,\lambda}$.*

We consider example 2.2 ie $K = [0,1]$, $\gamma_1(y) = \frac{1}{2}y$ and $\gamma_2(y) = \frac{1}{2}(y+1)$. In this case, $B(\gamma) = \phi$ and $C(\gamma) = \phi$. β-KMS state on $\mathcal{O}_\gamma$ exists if and only if $\lambda = \log 2$ and given by the normalized Lebesgue measure on $[0,1]$.

5.2. **Sierpinski Gasket.** As the case of dynamics on unit interval, we can classify KMS states for the C*-algebra associated with Sierpinski Gasket introduced in Kajiwara-Watatani [11]. The contractions in this example are considered to be cross sections for rational map on Riemaniann sphere whose Julia set is homeomorphic to Sierpinski Gasket.

Let Ω be a regular triangle in $\mathbf{R}^2$ with three vertexes $c_1 = (1/2, \sqrt{3}/2)$, $c_2 = (0,0)$ and $c_3 = (1,0)$. The middle point of c_1c_2 is denote by b_1, the middle point of c_1c_3 is denoted by b_2 and the middle point of c_2c_3 is denoted by b_3. We define proper contractions $\tilde{\gamma}_i$ $(i = 1,2,3)$ by

$$\tilde{\gamma}_1(x,y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right), \quad \tilde{\gamma}_2(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \tilde{\gamma}_3(x,y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right).$$

Let α_θ be a rotation by the angle θ. We put $\gamma_1 = \tilde{\gamma}_1$, $\gamma_2 = \alpha_{-2\pi/3} \circ \tilde{\gamma}_2$ and $\gamma_3 = \alpha_{2\pi/3} \circ \tilde{\gamma}_3$. We denote by S with the metric d induce from $\mathbf{R}^2$ the self similar set determined by $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. We note that c_i and b_i $i = 1, 2, 3$ are contained in S. Putting $V = S \backslash \{c_1, c_2, c_3\}$, γ satisfies the open set condition. In this case, we have $B(\gamma) = \{b_1, b_2, b_3\}$ and $C(\gamma) = \tilde{C}(\gamma) = \{c_1, c_2, c_3\}$, and γ satisfies the finite branch condition. We denote by $\mathcal{O}_{S,\gamma}$ Cuntz-Pimsner algebra constructed from S and the above γ.

Let μ be the Borel probability measure on (S, d) satisfying the condition (3) and (4). We get the conditions of point mass of μ.

Lemma 5.11. *If $\lambda > 1$, we have $c_\mu(c_1) = 0$, $c_\mu(c_2) = 0$ and $c_\mu(c_3) = 0$.*

Proof. We note that $\gamma^{-1}(c_1) = \{c_1\}$, $\gamma^{-1}(c_2) = \{c_3\}$ and $\gamma^{-1}(c_3) = \{c_2\}$. By Lemma 4.12, we have

$$c_\mu(c_1) = \lambda c_\mu(c_1) \qquad c_\mu(c_2) = \lambda c_\mu(c_3) \qquad c_\mu(c_3) = \lambda c_\mu(c_2).$$

When $\lambda > 1$, then these show that $c_\mu(c_1) = 0$, $c_\mu(c_2) = 0$ and $c_\mu(c_3) = 0$. □

As in the case of dynamics on unit interval, we have the following Lemmas.

Lemma 5.12. *For $y \in B(\gamma)$, we have $O(y) \cap C(\gamma) = \phi$, and for y, $y' \in B(\gamma)$ with $y \neq y'$, we have $O(y) \cap O(y') = \phi$.*

Lemma 5.13. *If μ satisfies the condition (3) and (4) in Theorem 4.7 and does not have a point mass at $B(\gamma)$, or if μ satisfies the condition (3) in Theorem 4.7 and does not have a point mass at $B(\gamma) \cup C(\gamma)$, we have $\lambda = 3$ and $\mu = \mu_H$.*

Lemma 5.14. *If μ satisfying the condition (3) and (4) in Theorem 4.7 has a point mass at $B(\gamma)$, then we have $\lambda > 3$ and $c_\mu(\gamma_{j_1} \cdots \gamma_{j_n}(y)) = \lambda^{-n} c_\mu(y)$.*

Lemma 5.15. *If μ has the condition (3) and (4) in Theorem 4.7, then μ does not have a point mass at $C(\gamma)$.*

Let $\lambda > 3$. As in dynamics on unit interval, for $y \in B(\gamma)$, then we define a probability measure $\mu_{\lambda,y}$ as follows:

$$\tau^{\mu_{y,\lambda}}(a) = \frac{\lambda - 3}{\lambda} \sum_{n=0}^{\infty} \sum_{(j_1,\dots,j_n) \in \{1,2,3\}^n} a(\gamma_{j_1} \cdots \gamma_{j_n}(y)).$$

As dynamic for unit interval, we have the following:

Lemma 5.16. *$\mu_{\lambda,y}$ satisfies the condition (3) and (4), and is extended to the $\log \lambda$-KMS state on $\mathcal{O}_{S,\gamma}$.*

We can get classification of KMS states on $\mathcal{O}_{S,\gamma}$. Let $\beta = \log \lambda$.

Theorem 5.17. *Let S be the Sierpinski gasket defined by contractions γ as above. Then β-KMS state on $\mathcal{O}_{S,\gamma}$ with respect to the gauge action exists only if $\lambda \geq 3$ and are classified as follows:*

(1) *When* $\lambda = 3$, φ^H *is the unique KMS state.*
(2) *When* $\lambda > 3$, *each* β*-KMS state is expressed by a convex combination of* $\{\varphi^{y,\lambda} \mid y = b_1, b_2, b_3\}$

Moreover $\varphi_{y,\lambda}$*'s are an extreme* $\log \lambda$*-KMS state.*

References

[1] Bratteli O. and Robinson D.W., *Operator algebras and quantum statistical mechanics II. Equilibrium states. Models in quantum statistical mechanics.,* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997

[2] Enomoto M., Fujii M. and Watatani Y., *KMS states for gauge action on* O_A*,* Math. Japon. 29(1984), 607–619

[3] Evans D., *On* O_n*,* Publ. Res. Inst. Math. Sci. Kyoto Univ. 16(1980)

[4] Exel R., *Crossed-products by finite index endomorphisms and KMS states,* J. Funct. Anal. 199(2003), 153–183

[5] Exel R., *KMS states for generalized gauge actions on Cuntz-Krieger algebras (An application of the Ruelle-Perron-Frobenius Theorem)* [arXiv : math.OA/0110183]

[6] Exel R. and Laca M., *Partial dynamical systems and the KMS condition,* Commun. Math. Phys., 232(2003), 223–277

[7] Fowler N.J., Muhly P.S. and Raeburn I., *Representations of Cuntz-Pimsner Algebras*, Indiana Univ. Math. J. **52** (2003), 569–605.

[8] Huchchinson J.E., *Fractals and self-similarity,* Indiana Univ. Math. J. 30 (1981), 713–747.

[9] Kajiwara T., Pinzari C. and Watatani Y., *Jones index theory for Hilbert* C^**–bimodules and its equivalence with conjugation theory,* J. Funct. Anal. in press

[10] Kajiwara T. and Watatani Y., C^**-algebra associated with complex dynamical systems,* [arXiv : math.OA/0309293]

[11] Kajiwara T. and Watatani Y., C^**-algebras associated with self-similar sets* [arXiv : math.OA/0312481]

[12] Kerr D. and Pinzari C., *Noncommutative pressure and the variational principle in Cuntz-Krieger-type* C^**-algebras.* J. Funct. Anal. 188(2002), 156–215

[13] Kumjian A. and Renalut J., *KMS states on* C^**-algebras associated to expansive maps,* [arXiv : math. OA/0305044]

[14] Laca M. and Neshveyev S., *KMS states of quasi-free dynamics on Pimsner algebras,* J. Funct. Anal. 211(2004) 457–482

[15] Matsumoto K., Watatani Y. and Yoshida M. *KMS-states for gauge action on* C^**-algebras associated with subshifts,* Math. Z. 228(1998), 489–509

[16] Melo W. and Strien S., *One-dimesional Dynmics,* Springer, 1993

[17] Muhly P.S. and Tomforde M., *Topological quivers,* [arXiv : math.OA/0312109]

[18] Olsen D. and Pederson G.K., *Some* C^**-dynamical systems with a single KMS state,* Math. Scand. 42(1978), 111–118

[19] Pimsner M., *A class of* C^**-algebras generating both Cuntz-Krieger algebras and crossed product by* $\mathbb{Z}$*,* Free probability theory, AMS, (1997), 189–212.

[20] Pinzari C., Watatani Y. and Yonetani K. *KMS states, entropy and the variational principle in full* C^* *-dynamical systems,* Commun. Math. Phys. 213(2000), 331–379

[21] Renault, J., *A groupoid approach to* C^**-algebras* Lecture Notes in Mathematics, 793. Springer, Berlin, 1980.

[22] Renault, J., *Cuntz-like algebras,* Operator theoretical methods (Timicsoara, 1998), (2000), 371–386, Theta Found., Bucharest

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